ELC 4351: Digital Signal Processing

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Frequency-domain Analysis of LTI Systems

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Frequency-Domain Characteristics of Linear Time-Invariant Systems

Response to Complex Exponential and Sinusoidal Signals: The Frequency Response Function $H(\omega)$

LTI system to an arbitrary input signal $x(n)$:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Input excitation is the complex exponential: $x(n) = Ae^{j\omega n}, -\infty < n < \infty$. The response is

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)} = A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n}$$

Frequency-Domain Characteristics of Linear Time-Invariant Systems

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) A e^{j\omega(n-k)} = A\left[\sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}\right] e^{j\omega n}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}$$

$$y(n) = AH(\omega) e^{j\omega n}$$

$x(n) = Ae^{j\omega n}$ is an eigenfunction of the system.

$H(\omega)$ is the corresponding eigenvalue of the system.

Frequency-Domain Characteristics of Linear Time-Invariant Systems

Since $H(\omega)$ is the Fourier transform of $\{h(k)\}$, it follows that $H(\omega)$ is a periodic function with period 2π .

Impulse Response and Frequency Response

$$
\begin{aligned}
H(\omega) &= \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \\
h(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{j\omega k} d\omega
\end{aligned}
$$

Frequency-Domain Characteristics of Linear Time-Invariant Systems

If $h(k)$ is a real-valued impulse response, then

$$H_R(\omega) = H_R(-\omega) \quad \text{and} \quad H_I(\omega) = -H_I(-\omega)$$

$$|H(\omega)| = |H(-\omega)| \quad \text{and} \quad \angle H(\omega) = -\angle H(-\omega)$$

Response to Aperiodic Input Signals

From the convolution theorem:

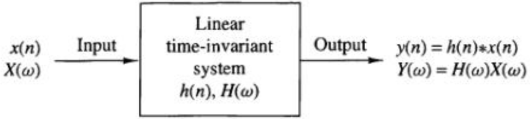
LTI systems

$$Y(\omega) = H(\omega)X(\omega)$$

$$\begin{aligned} |Y(\omega)| &= |H(\omega)||X(\omega)| \\ \angle Y(\omega) &= \angle H(\omega) + \angle X(\omega) \end{aligned}$$

LTI systems

$$Y(\omega) = H(\omega)X(\omega)$$



We observe that the output of a linear time-invariant system cannot contain frequency components that are not contained in the input signal. It takes either a linear time-variant system or a nonlinear system to create frequency components that are not necessarily contained in the input signal.

Response to Aperiodic Input Signals

LTI systems

$$Y(\omega) = H(\omega)X(\omega)$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

Frequency Response of LTI Systems

We focus on determining the frequency response of LTI systems that have rational system functions.

Recall that this class of LTI systems is described in the time domain by constant-coefficient difference equations.

Frequency Response of a System with a Rational System Function

If the system function $H(z)$ converges on the unit circle, we can obtain the frequency response of the system by evaluating $H(z)$ on the unit circle.

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

If $H(z)$ is a rational function, we have

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

Frequency Response of a System with a Rational System Function

$$
\begin{aligned}
H(\omega) &= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1+\sum_{k=1}^{N} a_k e^{-j\omega k}} \\
&= b_0 \frac{\prod_{k=1}^{M}(1-z_k e^{-j\omega})}{\prod_{k=1}^{N}(1-p_k e^{-j\omega})}
\end{aligned}
$$

$$
\begin{aligned}
H^*(\omega) &= b_0 \frac{\prod_{k=1}^{M}(1-z_k^* e^{j\omega})}{\prod_{k=1}^{N}(1-p_k^* e^{j\omega})} \\
H^*(1/z^*) &= b_0 \frac{\prod_{k=1}^{M}(1-z_k^* z)}{\prod_{k=1}^{N}(1-p_k^* z)}
\end{aligned}
$$

Frequency Response of a System with a Rational System Function

When $h(n)$ is real or, equivalently, the coefficients $\{a_k\}$ and $\{b_k\}$ are real, complex-valued poles and zeros occur in complex-conjugate pairs.

In this case, $H^*(1/z^*) = H(z^{-1})$. Consequently, $H^*(\omega) = H(-\omega)$, and

$$\boxed{|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H(z^{-1})|_{z=e^{j\omega}}}$$

- $H(z)H(z^{-1})$ is the z-transform of $r_{hh}(n)$.
- It follows that $|H(\omega)|^2$ is the Fourier transform of $r_{hh}(n)$.

Correlation Functions and Spectra at the Output of LTI Systems

$$
\begin{aligned}
r_{yy}(n) &= r_{hh}(n) \otimes r_{xx}(n) \\
r_{yx}(n) &= h(n) \otimes r_{xx}(n)
\end{aligned}
$$

Using z-transform:

$$
\begin{aligned}
S_{yy}(z) &= S_{hh}(z) S_{xx}(z) \\
S_{yx}(z) &= H(z) S_{xx}(z)
\end{aligned}
$$

Substitute $z = e^{j\omega}$, we have

$$
\begin{aligned}
S_{yy}(\omega) &= |H(\omega)|^2 S_{xx}(\omega) \\
S_{yx}(\omega) &= H(\omega) S_{xx}(\omega)
\end{aligned}
$$

Input-Output Correlation Functions and Spectra

The energy in the output signal:

$$
\begin{aligned}
E_y &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(\omega)d\omega = r_{yy}(0) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(\omega)|^2 S_{xx}(\omega)d\omega
\end{aligned}
$$

Correlation Functions and Power Spectra for Random Input Signals

The expected value of the output signal:

$$
\begin{aligned}
m_y &= \mathrm{E}[y(n)] = \mathrm{E}\left[\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right] \\
&= \sum_{k=-\infty}^{\infty} h(k)\mathrm{E}[x(n-k)] \\
&= m_x \sum_{k=-\infty}^{\infty} h(k) \\
&= m_x \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}|_{\omega=0} \\
&= m_x H(0)
\end{aligned}
$$

Correlation Functions and Power Spectra for Random Input Signals

The autocorrelation sequence of the output random process:

$$\begin{aligned}
\gamma_{yy}(m) &= \mathrm{E}[y^*(n)y(n+m)] \\
&= \mathrm{E}\left[\sum_{k=-\infty}^{\infty} h(k)x^*(n-k) \sum_{l=-\infty}^{\infty} h(l)x(n+m-l)\right] \\
&= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(k)h(l)\mathrm{E}[x^*(n-k)x(n+m-l)] \\
&= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(k)h(l)\gamma_{xx}(k-l+m)
\end{aligned}$$

Correlation Functions and Power Spectra for Random Input Signals

The power density spectrum of the output random process:

$$
\begin{aligned}
\Gamma_{yy}(\omega) &= \sum_{m=-\infty}^{\infty} \gamma_{yy}(m) e^{-j\omega m} \\
&= \sum_{m=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(k) h(l) \gamma_{xx}(k-l+m) \right] e^{-j\omega m} \\
&= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(k) h(l) \left[\sum_{m=-\infty}^{\infty} \gamma_{xx}(k-l+m) e^{-j\omega m} \right] \\
&= \Gamma_{xx}(\omega) \left[\sum_{k=-\infty}^{\infty} h(k) e^{j\omega k} \right] \left[\sum_{l=-\infty}^{\infty} h(l) e^{-j\omega l} \right] \\
&= |H(\omega)|^2 \Gamma_{xx}(\omega)
\end{aligned}
$$

Correlation Functions and Power Spectra for Random Input Signals

$$
\begin{aligned}
m_y &= m_x H(0) \\
\Gamma_{yy}(\omega) &= |H(\omega)|^2 \Gamma_{xx}(\omega) \\
\Gamma_{yx}(\omega) &= H(\omega) \Gamma_{xx}(\omega)
\end{aligned}
$$

Correlation Functions and Power Spectra for Random Input Signals

When the input random process is white, i.e.,

$$m_x = 0 \qquad \text{and} \qquad \gamma_{xx}(m) = \sigma_x^2 \delta(m)$$

$$\begin{aligned} m_y &= 0 \\ \Gamma_{xx}(\omega) &= \sigma_x^2 \\ \Gamma_{yy}(\omega) &= |H(\omega)|^2 \sigma_x^2 \\ \Gamma_{yx}(\omega) &= H(\omega) \sigma_x^2 \end{aligned}$$