ELC 4351: Digital Signal Processing

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The z-Transform and Its Application to the Analysis of LTI Systems

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Laplace-Transform: Continuous-time signals and LTI systems

z-Transform: Discrete-time signals and LTI systems

The Direct z-Transform

The direct z-transform is a power series.

Transform Equation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where, z is a complex variable.

It can be expressed as $X(z) = \mathcal{Z}\{x(n)\}$ or $x(n) \overset{z}{\longleftrightarrow} X(z)$.

The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

Discussion on ROC

$z = re^{j\theta}$. $r = |z|$ and $\theta = \angle z$.

Transformation Equation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

In the ROC, $|X(z)| < \infty$.
Therefore

$$\begin{aligned} |X(z)| &= |\sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}| \\ &\leq \sum_{n=-\infty}^{\infty} |x(n) r^{-n} e^{-j\theta n}| \\ &= \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| \end{aligned}$$

Discussion on ROC

$$|X(z)| \leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty$$

$|X(z)|$ is finite if the sequence $x(n)r^{-n}$ is absolutely summable.

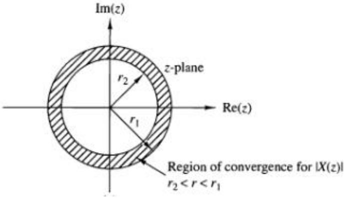
Discussion on ROC

$$
\begin{aligned}
|X(z)| &\leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| \\
&= \sum_{n=-\infty}^{-1} |x(n)r^{-n}| + \sum_{n=0}^{\infty} |x(n)r^{-n}| \\
&= \underbrace{\sum_{n=1}^{\infty} |x(-n)r^{n}|}_{\text{finite: r small enough}} + \underbrace{\sum_{n=0}^{\infty} \left|\frac{x(n)}{r^{n}}\right|}_{\text{finite: r large enough}}
\end{aligned}
$$

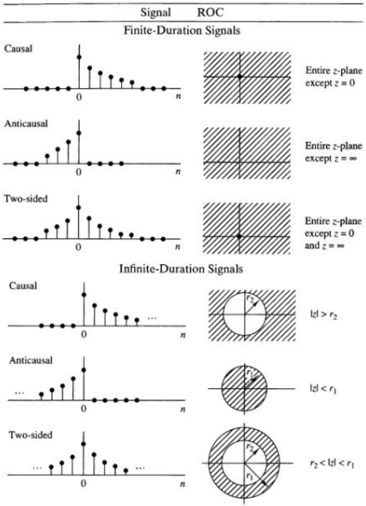
In general, ROC: $r_2 < r < r_1$

Discussion on ROC

ROC: $r_2 < r < r_1$



Discussion on ROC



Unilateral z-Transform

Transformation Equation

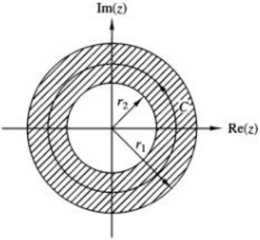
$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The Inverse z-Transform

Transformation Equation

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C denotes the closed contour in the ROC of $X(z)$, taken in a counterclockwise direction.



Properties of the z-Transform

Linearity

If $x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \stackrel{z}{\longleftrightarrow} X_2(z)$, then

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) \stackrel{z}{\longleftrightarrow} X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z)$$

for any constants α_1 and α_2.

Time shifting

If $x(n) \stackrel{z}{\longleftrightarrow} X(z)$, then

$$x(n-k) \stackrel{z}{\longleftrightarrow} z^{-k} X(z)$$

The ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$.

Properties of the z-Transform

Scaling in the z-domain

If $x(n) \overset{z}{\longleftrightarrow} X(z)$, ROC: $r_1 < |z| < r_2$, then

$$a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z), \qquad \text{ROC}: |a|r_1 < |z| < |a|r_2$$

for any constants a, real or complex.

Time reversal

If $x(n) \overset{z}{\longleftrightarrow} X(z)$, ROC: $r_1 < |z| < r_2$, then

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}), \qquad \text{ROC}: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Properties of the z-Transform

Differentiation in the z-domain

If $x(n) \overset{z}{\longleftrightarrow} X(z)$, then

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

Properties of the z-Transform

Convolution of two sequences

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$, then

$$x(n) = x_1(n) \otimes x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$

The ROC of $X(z)$ is at least the intersection of that for $X_1(z)$ and $X_2(z)$.

Correlation of two sequences

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$, then

$$r_{x_1 x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \overset{z}{\longleftrightarrow} R_{x_1 x_2}(z) = X_1(z)X_2(z^{-1})$$

The ROC of $R(z)$ is at least the intersection of that for $X_1(z)$ and $X_2(z^{-1})$.

Multiplication of two sequences

If $x_1(n) \stackrel{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \stackrel{z}{\longleftrightarrow} X_2(z)$, then

$$x(n) = x_1(n)x_2(n) \stackrel{z}{\longleftrightarrow} X(z) = \frac{1}{2\pi j} \oint_C X_1(\nu)X_2(\frac{z}{\nu})\nu^{-1}d\nu$$

where C is a closed contour that encloses the origin and lies within the ROC common to both $X_1(\nu)$ and $X_2(1/\nu)$.

Properties of the z-Transform

Parseval's relation

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(\nu)X_2^*(\frac{1}{\nu^*})\nu^{-1}d\nu$$

Properties of the z-Transform

The Initial Value Theorem

If $x(n)$ is causal, i.e. $x(n) = 0$ for $n < 0$, then

$$x(0) = \lim_{z\to\infty} X(z)$$

Proof.

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x(n)z^{-n} \\ &= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \end{aligned}$$

As $z \to \infty$, $z^{-n} \to 0$ when $n = 1, 2, \ldots$, therefore $X(z) \to x(0)$.