ELC 4351: Digital Signal Processing

Liang (Leon) Dong

Electrical and Computer Engineering
Baylor University

liang_dong@baylor.edu

z-Transform Part 2

The z-Transform and Its Application to the Analysis of LTI Systems

Rational z-Transform

Rational z-Transforms

$X(z)$ is a rational function, that is, a ratio of two polynomials in z^{-1} (or z).

$$
\begin{aligned}
X(z) &= \frac{B(z)}{A(z)} \\
&= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots a_N z^{-N}} \\
&= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}
\end{aligned}
$$

Rational z-Transforms

$X(z)$ is a rational function, that is, a ratio of two polynomials $B(z)$ and $A(z)$. The polynomials can be expressed in factored forms.

$$
\begin{aligned}
X(z) &= \frac{B(z)}{A(z)} \\
&= \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)} \\
&= \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}
\end{aligned}
$$

Poles and Zeros

The zeros of a z-transform $X(z)$ are the values of z for which $X(z) = 0$.

The poles of a z-transform $X(z)$ are the values of z for which $X(z) = \infty$.

$$X(z) \quad = \quad \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

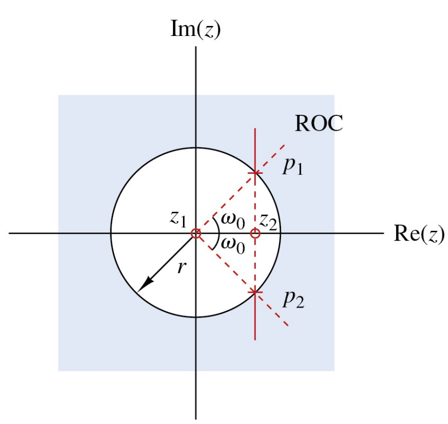
$X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$, N finite poles at $z = p_1, p_2, \ldots, p_N$, and $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin.

Poles and zeros may also occur at $z = \infty$.

$X(z)$ has exactly the same number of poles and zeros.

Poles and Zeros

If a polynomial has real coefficients, its roots are either real or occur in complex-conjugate pairs. That is because e.g., $(z - p_1)(z - p_2)$

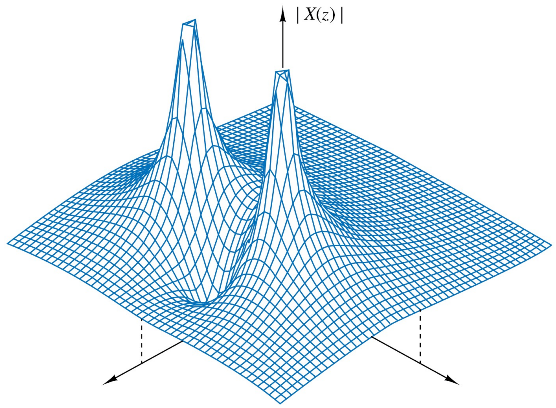


Poles and Zeros

For example,

$$X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.2732z^{-1} + 0.81z^{-2}}$$

which has one zero at $z = 1$ and two poles at $p_1 = 0.9e^{j\pi/4}$ and $p_2 = 0.9e^{-j\pi/4}$.



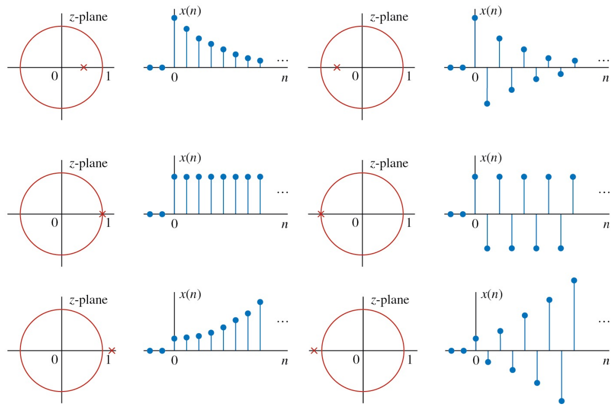
Some Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Poles Locations and Time-Domain Behavior for Causal Signals

If a real signal has a z-transform with one pole, this pole has to be real. The only such signal is the real exponential

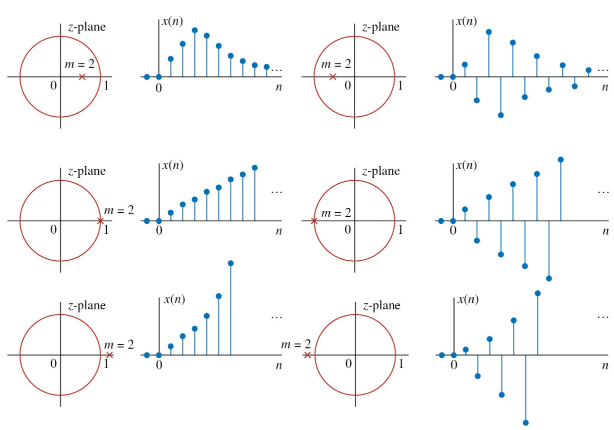
$$x(n) = a^n u(n) \rightarrow^z X(z) = \frac{1}{1 - az^{-1}}, \ \text{ROC} : |z| > |a|$$



Poles Locations and Time-Domain Behavior for Causal Signals

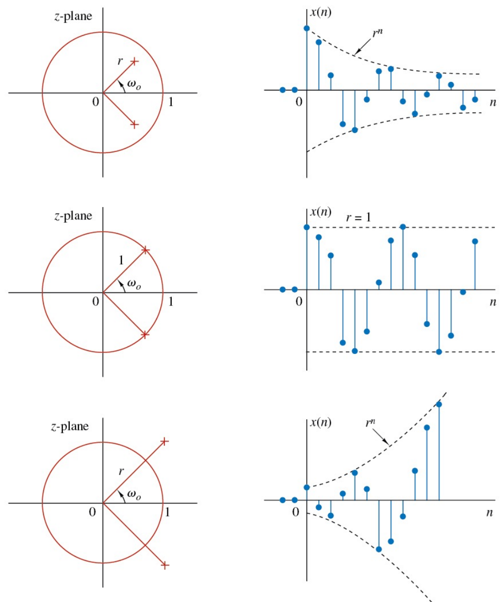
A causal real signal with a double real pole has the form

$$x(n) = na^n u(n) \rightarrow^z X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}, \ \text{ROC} : |z| > |a|$$



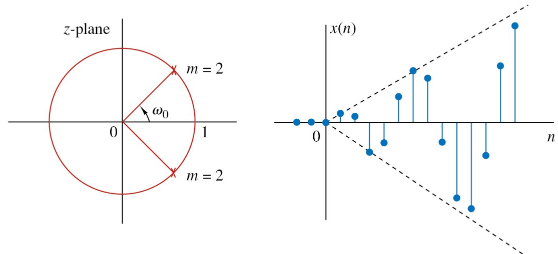
Poles Locations and Time-Domain Behavior for Causal Signals

The case of a causal signal with a pair of complex-conjugate poles.



Poles Locations and Time-Domain Behavior for Causal Signals

The case of a causal signal with a double pair of poles on the unit circle.



Poles Locations and Time-Domain Behavior for Causal Signals

The impulse response $h(n)$ of a causal LTI system is a causal signal.

If a pole of a system is outside the unit circle, the impulse response of the system becomes unbounded and, consequently, the system is unstable.

System Function of a LTI System

LTI systems:

$$\begin{aligned} y(n) &= h(n) \otimes x(n) \\ Y(z) &= H(z)X(z) \end{aligned}$$

If we know the input $x(n)$ and observe the output $y(n)$ of the system, we can determine the unit sample response (impulse response) by first solving for $H(z)$ from

$$H(z) = \frac{Y(z)}{X(z)}$$

and then evaluating the inverse z-transform of $H(z)$.

$H(z)$ is called the system function.

System Function of a LTI System

When the LTI system is described by a linear constant-coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

The system function can be calculate:

$$\begin{aligned} Y(z) &= -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k} \\ Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) &= X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \end{aligned}$$

System Function of a LTI System

An LTI system described by a constant-coefficient difference equation has a rational system function $H(z)$.

$$\boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}}$$

System Function of a LTI System

(1) All-zero system: If $a_k = 0$ for $1 \leq k \leq N$,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

The system has M nontrivial zeros and M trivial poles (at $z = 0$).

An all-zero system is an FIR system and can be called a moving average (MA) system.

System Function of a LTI System

(2) All-pole system: If $b_k = 0$ for $1 \leq k \leq M$,

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{M} a_k z^{N-k}}$$

where $a_0 = 1$. The system has N nontrivial poles and N trivial zeros (at $z = 0$).

An all-pole system is an IIR system and can be called an auto-regressive (AR) system.

(3) Pole-zero system:

In general, the system function contains N poles and M zeros. (Poles and zeros at $z = 0$ and $z = \infty$ are implied but are not counted explicitly.)

Due to the presence of poles, a pole-zero system is an IIR system.